B.A/B.Sc 6th Semester (Honours) Special Examination, 2020 (CBCS)
Subject: Mathematics
Course: BMH6 CC 13
(Metric Spaces and Complex Analysis)

Time:3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: **6×5 = 30**

(a) Prove that the sequence space $l_p (1 \le p < \infty)$ is a complete metric space with respect to suitable metric to be defined by you.

(b) Let (X, d) and (Y, ρ) be two metric spaces. Prove that $f : X \to Y$ is continuous if and only if $\overline{(f^{-1}(B))} \subset f^{-1}(\overline{B})$ for every $B \subset Y$.

(c) Prove that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

(d) Prove that the intersection of any collection of complete subsets of a metric space is complete and the union of a finite collection of complete subsets of a metric space is complete.

(e) Prove that in a metric space, if a connected set is contained in the union of two separated sets then it is contained in exactly one of them.

(f) Prove that the function $f(z) = u + iv$, where $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} (z \neq 0), f(0) = 0$ is continuous. Also prove that Cauchy-Riemann equations are satisfied at the origin, but $f'(z)$ does not exist there.

(g) Expand the function $f(z) = \dfrac{2z^3 + 1}{z^2 + z}$ as a Laurent series within the annulus $0 < |z| < 2$.

(h) Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^4 + a^4} (a > 0)$.

2. Answer any three questions: **3×10 =30**

(a) (i) Prove that in a metric space, the continuous image of a connected set is connected.

(ii) Let $f, g : [0,1] \to \mathbb{R}$ be continuous. Let $f(x) \in [0,1]$ for all $x \in [0,1]$, $g(0) = 0$ and $g(1) = 1$. Show that $f(x) = g(x)$ for some $x \in [0,1]$.

(iii) Prove that in a metric space, arbitrary union of connected sets with non-empty intersection is connected. 3+3+4

(b) (i) State and prove Heine-Borel theorem on compactness in a metric space.

(ii) Prove that every totally bounded subset of a metric space is bounded. Give an example to show that the converse is not always true. (1+4)+(3+2)

(c) (i) Let $f:[a,b]\to\mathbb{R}$ be continuous. Show that the graph of f is a compact subset of $\mathbb{R}^2$.

(ii) If A is a compact subset in a metric space (X,d) with diameter $\delta(A)$, prove that $\exists\, x,y\in A$ such that $d(x,y)=\delta(A)$.

(iii) Let (X,d) be a compact metric space and let $f:X\to X$ be such that $d(f(x),f(y))<d(x,y)\;\forall\, x,y\in X$ with $x\neq y$. Show that f has a fixed point. Is the fixed point unique? 3+3+4

(d) (i) If $f(z)$ is an analytic function of z, prove that $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$.

(ii) If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $u-v=e^x(\cos y-\sin y)$, find $f(z)$ in terms of z.

(iii) Find the radius of convergence of the power series $f(z)=\sum_0^\infty \dfrac{z^n}{2^n+1}$ and prove that $(2-z)f(z)-2\to 0$ as $z\to 2$. 4+3+(1+2)

(e) (i) By contour integration prove that $\displaystyle\int_0^\infty \frac{\log(1+x^2)}{1+x^2}dx=\pi\log 2$.

(ii) Prove that $\displaystyle\int_\nu \frac{dz}{z-a}=2\pi i$

where ν is given by the equation $|z-a|=R$.

(iii) Define $\sin z$ and $\cos z$ in $\mathbb{C}$ and prove that $\sin^2 z+\cos^2 z=1$. 5+2+(1+2)