B.A/B.Sc 6th Semester (Honours) Special Examination, 2020 (CBCS)
Subject: Mathematics
Course: BMH6CC14
(Ring Theory and Linear Algebra-II)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: **6×5=30**

(a) Prove that the ring $\mathbb{Z}[i]$ of Gaussian integer is PID.
(b) Prove that every Euclidean domain is a PID.
(c) Prove that every PID is UFD.
(d) Prove that an integral domain is a field if and only if there are exactly two associate classes.

(e) Let V be a vector space over the field F. If f and g are in V^* such that $f(v)=0$ implies $g(v)=0$, prove that $g=\beta f$ for some β in F.

(f) Let I be non-zero ideal of ring $\mathbb{Z}[i]$. Then prove that the quotient ring $\mathbb{Z}[i]/I$ is finite.

(g) Let $P(\mathbb{R})$ be the inner product space of all polynomials with real coefficients over the field $\mathbb{R}$ with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$ and $P_3(\mathbb{R})$ be the subspace of $P(\mathbb{R})$ consisting of all polynomials of degree less than 3. Using Gram – Schmidt process replace the standard basis $\{1, x, x^2\}$ of $P_3(\mathbb{R})$ by an orthonormal basis of $P_3(\mathbb{R})$.
(h) If W is a subspace of an inner product space V over the field F and if $v \in V$ satisfies $\langle v, w\rangle + \langle w, v\rangle \leq \langle w, w\rangle$ for every $w \in W$, prove that $\langle v, w\rangle = 0$ for all $w \in W$.

2. Answer any three questions. **3×10=30**

(a) (i) Prove that a polynomial f over a field R is a unit in $R[x]$ if and only if f is non-constant polynomial.
(ii) Prove that $\mathbb{Z}_n[x]$ is an integral domain if and only if n is a prime number.

(b) (i) Let R be a PID. Then prove that any non-zero proper ideal of ring R can be expressed as a finite product of maximal ideals of R.

(ii) Let R be a UFD and P be a non-zero prime ideal of R. Then prove that there exists an irreducible element in P.

(c) Let $V(\mathbb{R})$ be the vector space of all polynomial functions p from $\mathbb{R}$ to $\mathbb{R}$ which have degree less than or equal to 2. Let t_1, t_2, t_3 be three distinct real numbers and L_i: $V \longrightarrow \mathbb{R}$ be such that $L_i(p(x))=p(t_i)$, i=1,2,3. Show that $\{L_1, L_2, L_3\}$ is a basis of V^*. Determine a basis for V such that $\{L_1,L_2,L_3\}$ is its dual.

(d) (i) Let $f(x)= 5+11x-7x^2+9x^3$ in $\mathbb{Z}[x]$. Prove that $f(x)$ is irreducible over $\mathbb{Q}$ as well as over $\mathbb{Z}$.

(ii) In the inner product space $\mathbb{R}^3(\mathbb{R})$, show that the Schwarz inequality implies that the absolute value of cosine of an angle is at most 1.

(e) (i) Determine all irreducible polynomials of degree 2 in $\mathbb{Z}_2[x]$.

(ii) For the vectors u and v in an inner product space V over the field F, show that $|\langle u,v\rangle| = \|u\| \, \|v\|$ if and only if u and v are linearly dependent in V.